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An approximate theory for geometrically nonlinear thin plates

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Abstract

An approximate theory of thin plates is developed that is based on an assumed displacement field, the strains described by a Taylor series in the normal distance from the middle surface, the exact strains of the middle surface, and the equations of equilibrium governing the exact configuration of the deformed middle surface. In this theory, the exact geometry of the deformed middle surface is used to derive the strains and equilibrium of plates. This theory reduces to some existing nonlinear theories through imposition of constraints. Application of this theory does not depend on the constitutive law because, the physical deformation measure is used. It can also be applied when the response, loading and geometry of the plate are asymmetric and when the vibration mode number is not small. Predictions of membrane forces in a rectangular plate and of the equilibrium solution in a rotating disk are sample problems. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

A nonlinear theory for thin rotating disks and translating plates is of interest because of numerous applications to the modeling of: saws blades (Mote, 1965a), computer memory disks (Luo and Mote, 1999), translating bands (Mote, 1965b; Wang and Mote, 1994), turbine disks (Campbell, 1924), and others. At high speeds of common applications, the transverse vibration of these plates can be of large amplitude and highly nonlinear.

von Karman (1910), for at the first time presented a nonlinear plate theory when the nonlinear stretch effects in the transverse, equilibrium balance were considered. Love (1944), improved the equilibrium balance in the von Karman theory of 1910, when the curvature based on the nonlinear strain used by von Karman was used, and Chien (1944a,b) presented an intrinsic theory of shells through differential geometry. In the same year, Reissner (1944) introduced the deformation caused by shear strain into the bending of elastic plates through an assumed displacement field. Then, following the von Karman theory, Reissner (1957) presented his nonlinear plate theory that included shear deformation. Herrmann (1955) derived a plate theory governing dynamic motion with small elongation and shear deformation, but

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moderately large rotation. In the aforementioned references, the strain of the middle surface, approximated by either the linear terms of its Taylor series expansion about the unstrained middle surface of thin plates or the Lagrangian and Eulerian strains of the deformed middle surface was used. The in-plane equilibrium balances are referred in the undeformed middle surface. In nonlinear plate theories, the von Karman equations have been used to investigate the nonlinear vibration of a spinning disk (Nowinski, 1964, 1981; Renshaw and Mote, 1995; Hamidzadeh et al., 1998). The von Karman equations considered the nonlinear effects only in the transverse direction, and such equations did not include the Coriolis acceleration.

In 1990, Wang developed a two-dimensional theory reduced from the three-dimensional theory for a transversely isotropic body through the decomposition of general displacements in the three-dimensional body into the asymmetrical and symmetrical deformations, and Wang discussed the static plate problem rather than the dynamic plate theory. Hodges et al. (1993) developed the geometrically nonlinear plate theory through the introduction of warping displacement, and extra equations were presented for determining the warping displacement.

For a complete consideration of geometrical nonlinearity in plates, a displacement field at any material point is expressed by a series form of displacements in the middle surface and a normal distance from the middle surface in the transverse direction. The series expression is different from the displacement decomposition in Wang (1990). It also differs from the director series expression of displacements given by Naghdi (1972). In such a director expression, the infinite directors are unknown. However, the series expression of displacements proposed in this article can be determined through the plate assumptions, such as Kirchhoff assumptions, etc. From the assumed displacement field, an approximate theory of geometrically nonlinear plates will be developed herein through the nonlinear deformation theory in a 3-D body.

In this article, the physical strain and equilibrium equations in the plates will be derived based on the exact geometry of the deformed middle surfaces. This theory reduces to some established, approximate, nonlinear theories of the thin plates through imposition of constraints. Predictions of membrane forces in a rectangular plate and of the equilibrium solution in a rotating disk will be presented as sample problems.

2. A nonlinear theory of thin plates

2.1. Deformation of a 3-D body

Consider a material particle $P(Y^1, Y^2, Y^3)$ in a flexible body $\mathscr{B}_0$ at the initial state as shown in Fig. 1. The position $\mathbf{R}$ of the particle is described by Y^K:

$$\mathbf{R} = Y^K\mathbf{I}_K \equiv Y^1\mathbf{I}_1 + Y^2\mathbf{I}_2 + Y^3\mathbf{I}_3, \tag{1}$$

where $\mathbf{I}_K$ are unit vectors in the fixed coordinates. In the local curvilinear reference frame, $\mathbf{R}$ is represented by

$$\mathbf{R} = X^K\mathbf{G}_K \equiv X^1\mathbf{G}_1 + X^2\mathbf{G}_2 + X^3\mathbf{G}_3, \tag{2}$$

where the component $X^K = \mathbf{R}\cdot\mathbf{G}^K$ in Eringen (1962) and the initial base vectors $\mathbf{G}_K \equiv \mathbf{G}_K(X^1, X^2, X^3, t_0)$ are

$$\mathbf{G}_K = \frac{\partial\mathbf{R}}{\partial X^K} = \frac{\partial Y^M(X^1, X^2, X^3, t_0)}{\partial X^K}\mathbf{I}_M = Y^M_{,K}\mathbf{I}_M \tag{3}$$

with magnitudes

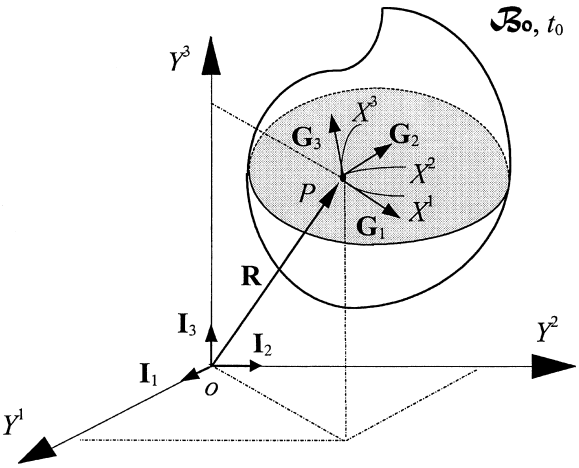

Fig. 1. A material particle P.

$$|\mathbf{G}_K(x)| = \sqrt{G_{KK}} = \sqrt{\left(\frac{\partial Y^1}{\partial X^K}\right)^2 + \left(\frac{\partial Y^2}{\partial X^K}\right)^2 + \left(\frac{\partial Y^3}{\partial X^K}\right)^2}\,\Bigg|_{t=t_0} \qquad \text{(no summation on } K) \tag{4}$$

and $G_{KL} = \mathbf{G}_K \cdot \mathbf{G}_L$ are metric coefficients in the body $\mathscr{B}_0$.

On deformation of $\mathscr{B}_0$, the particle at point P moves through displacement $\mathbf{u}$ to position p, and the particle Q, infinitesimally close to $P(X^1,X^2,X^3,t_0)$, moves through $\mathbf{u}+\mathrm{d}\mathbf{u}$ to q in the neighborhood of $p(X^1,X^2,X^3,t)$, as illustrated in Fig. 2.

The position of point p is

$$\mathbf{r} = \mathbf{R} + \mathbf{u} = (X^K + u^K)\mathbf{G}_K, \tag{5}$$

where the displacement is $\mathbf{u} = u^K\mathbf{G}_K$. Thus, $\overline{PQ} = \mathrm{d}\mathbf{R}$ and $\overline{pq} = \mathrm{d}\mathbf{r}$ are

$$\mathrm{d}\mathbf{R} = \mathbf{G}_K\,\mathrm{d}X^K, \qquad \mathrm{d}\mathbf{r} = \mathbf{G}_K\,\mathrm{d}X^K + \mathrm{d}\mathbf{u} \tag{6}$$

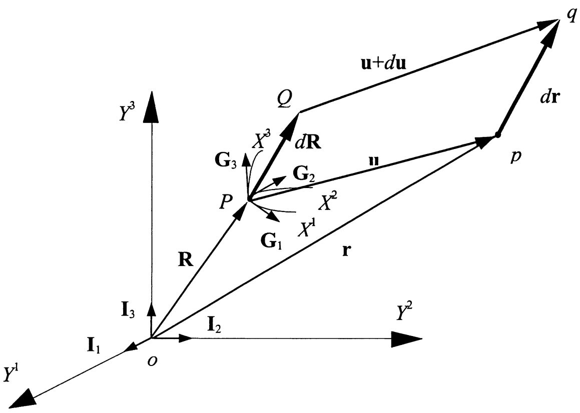

Fig. 2. Deformation of a differential linear element.

and the displacement becomes

$$\mathrm{d}\mathbf{u} = u^K_{;L}\,\mathrm{d}X^L\,\mathbf{G}_K = u_{K;L}\,\mathrm{d}X^L\,\mathbf{G}^K, \tag{7}$$

where $u^K_{;L} = \partial u^K/\partial X^L + {}_{t_0}\Gamma^K_{ML}u^M$ and ${}_{t_0}\Gamma^K_{ML}$ is the Christoffel symbol in Eringen (1967). The semicolon represents covariant partial differentiation. From Eqs. (6) and (7), we obtain

$$\mathrm{d}\mathbf{r} = \left(u^K_{;L} + \delta^K_L\right)\mathbf{G}_K\,\mathrm{d}X^L. \tag{8}$$

The Lagrangian strain tensor E_{LN} referred to the initial configuration is from Eringen (1962).

$$E_{\mathrm{LN}} = \tfrac{1}{2}\left(u_{L;N} + u_{N;L} + u^K_{;L}u_{K;N}\right). \tag{9}$$

As in Eringen (1962) (also see, Malvern, 1969), the change in length of d**R** per unit length gives

$$\underset{K}{\varepsilon_{\mathrm{N}}} = \frac{\left|\underset{K}{\mathrm{d}\mathbf{r}}\right| - \left|\underset{K}{\mathrm{d}\mathbf{R}}\right|}{\left|\underset{K}{\mathrm{d}\mathbf{R}}\right|} = \sqrt{1 + 2\frac{E_{KK}}{G_{KK}}} - 1, \tag{10}$$

where $\underset{K}{\varepsilon_{\mathrm{N}}}$ is the relative elongation along $\mathbf{G}_K$. The unit vectors along d**R** and d**r** in Eringen (1967) are

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{R}}{|\mathrm{d}\mathbf{R}|} = \frac{1}{\sqrt{G_{KK}}}\mathbf{G}_K, \qquad \mathbf{n} = \frac{\mathrm{d}\mathbf{r}}{|\mathrm{d}\mathbf{r}|} = \frac{(u^K_{;L} + \delta^K_L)}{\sqrt{G_{KK} + 2E_{KK}}}\mathbf{G}_K. \tag{11}$$

Accordingly, unit vectors of the deformed configuration in the directions $\mathbf{g}_1$ and $\mathbf{g}_2$ are

$$\underset{1}{\mathbf{n}} = \frac{\underset{1}{\mathrm{d}\mathbf{r}}}{\left|\underset{1}{\mathrm{d}\mathbf{r}}\right|}, \qquad \underset{2}{\mathbf{n}} = \frac{\underset{2}{\mathrm{d}\mathbf{r}}}{\left|\underset{2}{\mathrm{d}\mathbf{r}}\right|}. \tag{12}$$

Let Θ_{12} and θ_{12} be the induced angles between $\underset{1}{\mathbf{n}}$ and $\underset{2}{\mathbf{n}}$ before and after deformation. Then

$$\left.\begin{aligned} \cos\theta_{12} &= \cos(\Theta_{12} - \gamma_{12}) = \frac{\underset{1}{\mathrm{d}\mathbf{r}}\cdot\underset{2}{\mathrm{d}\mathbf{r}}}{\left|\underset{1}{\mathrm{d}\mathbf{r}}\right|\left|\underset{2}{\mathrm{d}\mathbf{r}}\right|} = \frac{G_{12} + 2E_{12}}{\sqrt{(G_{11} + 2E_{11})(G_{22} + 2E_{22})}}, \\ \cos\Theta_{12} &= \frac{\underset{1}{\mathrm{d}\mathbf{R}}\cdot\underset{2}{\mathrm{d}\mathbf{R}}}{\left|\underset{1}{\mathrm{d}\mathbf{R}}\right|\left|\underset{2}{\mathrm{d}\mathbf{R}}\right|} = \frac{G_{12}}{\sqrt{G_{11}G_{22}}} \end{aligned}\right\} \tag{13}$$

and the shear strain is

$$\gamma_{12} = \Theta_{12} - \theta_{12} = \cos^{-1}\frac{G_{12}}{\sqrt{G_{11}G_{22}}} - \cos^{-1}\left(\frac{G_{12} + 2E_{12}}{\sqrt{(G_{11} + 2E_{11})(G_{22} + 2E_{22})}}\right). \tag{14}$$

The other shear strains are obtained in a similar manner.

From Eq. (13), the direction cosine of the rotation is

$$\cos\left(\underset{K}{\mathbf{N}}, \underset{L}{\mathbf{n}}\right) = \frac{\underset{K}{\mathrm{d}\mathbf{R}}\cdot\underset{L}{\mathrm{d}\mathbf{r}}}{\left|\underset{K}{\mathrm{d}\mathbf{R}}\right|\left|\underset{L}{\mathrm{d}\mathbf{r}}\right|} = \frac{\delta^L_K + u^L_{;K}}{\sqrt{1 + \frac{E_{KK}}{G_{KK}}}} \quad (\text{no summation on } K). \tag{15}$$

In addition, the area and volume changes are given by

$$\frac{\underset{IJ}{\mathrm{d}a}}{\underset{IJ}{\mathrm{d}A}}=\frac{\left(1+\underset{I}{\varepsilon_{\mathrm{N}}}\right)\left(1+\underset{J}{\varepsilon_{\mathrm{N}}}\right)\sin\theta_{IJ}}{\sin\Theta_{IJ}},\qquad \frac{\mathrm{d}v}{\mathrm{d}V}=\left|\delta^I_J+u^I_{;J}\right|, \tag{16}$$

where $\mathrm{d}v$, $\mathrm{d}V$ are the differential material volumes after and before deformation, respectively; $|\cdot|$ represents the determinant. The areas after and before deformation are $\underset{IJ}{\mathrm{d}a}=\underset{I}{\mathrm{d}\mathbf{r}}\times\underset{J}{\mathrm{d}\mathbf{r}}$ and $\underset{IJ}{\mathrm{d}A}=\underset{I}{\mathrm{d}\mathbf{R}}\times\underset{J}{\mathrm{d}\mathbf{R}}$ where $I\neq J$.

2.2. Strain in thin plates

Let the Lagrangian coordinates be a rectangular Cartesian system:

$$Y^1=X^1=x,\quad Y^2=X^2=y,\quad Y^3=X^3=z \tag{17}$$

and

$$\mathbf{u}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}, \tag{18}$$

where $\mathbf{I}_1=\mathbf{i}$, $\mathbf{I}_2=\mathbf{j}$ and $\mathbf{I}_3=\mathbf{k}$. Therefore, from Eqs. (17) and (18), we have

$$G_{KK}=1,\quad G_{KL}=0,\quad \Gamma^K_{ML}=0,\quad \Theta_{KL}=\frac{\pi}{2}\quad (K\neq L). \tag{19}$$

Then, the physical strains in Eqs. (10) and (14) become

$$\varepsilon_I=\sqrt{\left(\delta^K_I+u_{K,I}\right)\left(\delta^K_I+u_{K,I}\right)}-1\qquad(\text{summation on }K), \tag{20}$$

$$\gamma_{IJ}=\sin^{-1}\left(\frac{\left(\delta^K_I+u_{K,I}\right)\left(\delta^K_J+u_{K,J}\right)}{\sqrt{\left(\delta^K_I+u_{K,I}\right)\left(\delta^K_I+u_{K,I}\right)}\sqrt{\left(\delta^K_J+u_{K,J}\right)\left(\delta^K_J+u_{K,J}\right)}}\right)\qquad(\text{summation on }K), \tag{21}$$

where $\{I,J,K\}=\{1,2,3\}$, $\{1\equiv x,\ 2\equiv y,\ 3\equiv z\}$ and $\{u_1\equiv u,\ u_2\equiv v,\ u_3\equiv w\}$. For reduction of the three-dimensional body displacements to a two-dimensional body form, displacements can be expressed in a Taylor series expanded about the displacement of the middle surface. Thus, similar to the basic kinematics hypothesis in Wempner (1973), the displacement field is represented by

$$u_I=u_I^{(0)}(x,y,t)+\sum_{n=1}^{\infty}z^n\varphi_I^{(n)}(x,y,t), \tag{22}$$

where $u_I^{(0)}$ denotes displacements of the middle surface, and the $\varphi_I^{(n)}(n=1,2,\ldots)$ are relative rotations. Substitution of Eq. (22) into Eqs. (20) and (21) and collection of like powers of z gives

$$\varepsilon_\alpha\approx\varepsilon_\alpha^{(0)}+\frac{\left(\delta^\alpha_I+u^{(0)}_{I,\alpha}\right)\varphi^{(1)}_{I,\alpha}}{\left(1+\varepsilon^{(0)}_\alpha\right)}z+\left\{\frac{2\left[\left(\delta^\alpha_I+u^{(0)}_{I,\alpha}\right)\varphi^{(2)}_{I,\alpha}\right]+\varphi^{(1)}_{I,\alpha}\varphi^{(1)}_{I,\alpha}}{1+\varepsilon^{(0)}_\alpha}-\frac{\left[\left(\delta^\alpha_I+u^{(0)}_{I,\alpha}\right)\varphi^{(1)}_{I,\alpha}\right]^2}{\left(1+\varepsilon^{(0)}_\alpha\right)^3}\right\}z^2+\cdots, \tag{23}$$

$$\varepsilon_3\approx\varepsilon_3^{(0)}+\frac{\left(\delta^I_3+\varphi^{(1)}_I\right)\varphi^{(2)}_I}{1+\varepsilon^{(0)}_\alpha}z+\left\{\frac{\left(4\varphi^{(2)}_I\varphi^{(2)}_I+6\left(\delta^I_3+\varphi^{(1)}_I\right)\varphi^{(3)}_I\right)}{1+\varepsilon^{(0)}_3}-\frac{\left[\left(\delta^I_3+\varphi^{(1)}_I\right)\varphi^{(2)}_I\right]^2}{\left(1+\varepsilon^{(0)}_3\right)^3}\right\}z^2+\cdots, \tag{24}$$

$$\gamma_{12} \approx \gamma_{12}^{(0)} + \frac{1}{\cos\gamma_{12}^{(0)}}\left\{\frac{\left(\delta_I^1 + \varphi_{I,1}^{(1)}\right)\left(\delta_I^2 + \varphi_{I,2}^{(1)}\right)}{\left(1+\varepsilon_1^{(0)}\right)\left(1+\varepsilon_2^{(0)}\right)} - \sin\gamma_{12}^{(0)}\left[\frac{\left(\delta_I^1 + u_{I,1}^{(0)}\right)\varphi_{I,1}^{(1)}}{\left(1+\varepsilon_1^{(0)}\right)^2} + \frac{\left(\delta_I^2 + u_{I,2}^{(0)}\right)\varphi_{I,2}^{(1)}}{\left(1+\varepsilon_2^{(0)}\right)^2}\right]\right\}z + \cdots, \tag{25}$$

$$\gamma_{\alpha 3} \approx \gamma_{\alpha 3}^{(0)} + \frac{1}{\cos\gamma_{\alpha 3}^{(0)}}\left\{\frac{\left(\delta_I^\alpha + \varphi_{I,\alpha}^{(1)}\right)\left(\delta_I^3 + \varphi_I^{(1)}\right)}{\left(1+\varepsilon_\alpha^{(0)}\right)\left(1+\varepsilon_3^{(0)}\right)} - \sin\gamma_{\alpha 3}^{0}\left[\frac{\left(\delta_I^\alpha + u_{I,\alpha}^{(0)}\right)\varphi_{I,\alpha}^{(1)}}{\left(1+\varepsilon_\alpha^{(0)}\right)^2} + \frac{2\left(\delta_I^3 + \varphi_I^{(1)}\right)\varphi_I^{(2)}}{\left(1+\varepsilon_3^{(0)}\right)^2}\right]\right\}z + \cdots, \tag{26}$$

where $\alpha = \{1, 2\}$. The strains of the middle surface, following Eqs. (20)–(22), at $z = 0$, are

$$\varepsilon_\alpha^{(0)} = \sqrt{\left(\delta_\alpha^K + u_{K,\alpha}^{(0)}\right)\left(\delta_\alpha^K + u_{K,\alpha}^{(0)}\right)} - 1 \quad \text{and} \quad \varepsilon_3^{(0)} = \sqrt{\left(\delta_3^K + \varphi_K^{(0)}\right)\left(\delta_3^K + \varphi_K^{(0)}\right)} - 1, \tag{27}$$

$$\gamma_{12}^{(0)} = \sin^{-1}\left(\frac{\left(\delta_1^K + u_{K,1}^{(0)}\right)\left(\delta_2^K + u_{K,2}^{(0)}\right)}{\left(1+\varepsilon_1^{(0)}\right)\left(1+\varepsilon_2^{(0)}\right)}\right) \quad \text{and} \quad \gamma_{\alpha 3}^{(0)} = \sin^{-1}\left(\frac{\left(\delta_\alpha^K + u_{K,\alpha}^{(0)}\right)\left(\delta_3^K + \varphi_K^{(0)}\right)}{\left(1+\varepsilon_\alpha^{(0)}\right)\left(1+\varepsilon_3^{(0)}\right)}\right). \tag{28}$$

In Eqs. (23)–(26), prediction of strain requires specification of three constraints for determination of the three sets $\varphi_I^{(n)} (I = 1, 2, 3; n = 1, 2, \ldots)$, like the assumptions $(\gamma_{\alpha 3} = \varepsilon_3 = 0)$ as in Kirchhoff (1850).

2.3. Equations of equilibrium for thin plates

Consider a thin plate subjected to the inertial force $\rho u_{I,tt}$, where $\rho = \int_{-h^-}^{h^+} \rho_0 \, \mathrm{d}z$ and ρ_0 is the density of the plate; body force $\mathbf{f} = \{f_I\}$; surface loading $\{p_I^+, p_I^-\}$, where the superscripts + and − denote the upper and lower surfaces; external moment m_α^0 before deformation. The distributed loading $\mathbf{q} = \{q_I\}$ has the components

$$q_I = p_I^+ - p_I^- + \int_{-h^-}^{h^+} f_I \, \mathrm{d}z \quad \text{and} \quad m_\alpha = m_\alpha^0 + h^+ p_\alpha^+ + h^- p_\alpha^- + \int_{-h^-}^{h^+} f_\alpha z \, \mathrm{d}z, \tag{29}$$

where $h = h^+ + h^-$. From Eqs. (14) and (16), the distributed loading after deformation becomes

$$\{q_I^*\} = \{q_I\}/[(1+\varepsilon_1)(1+\varepsilon_2)\cos\gamma_{12}]. \tag{30}$$

As the physical deformation measure is used in the Lagrangian coordinates, the constitutive laws based on such measure are

$$\sigma_{IJ} = F(\Lambda_{IJMN}, \varepsilon_{MN}, t) \tag{31}$$

used for the determination of the physical stresses directly (or sometime, termed the Kirchhoff stress), where Λ_{IJMN} is material properties (e.g.,Young's modulus, Poisson's ratio). Eq. (31) can be either Hooke's law for linear elastic materials or other similar laws for plasticity and others. Based on such physical stresses, the internal, resultant forces and moments can be determined in the Lagrangian coordinates. As in Wempner (1973), the use of Eqs. (16) and (31) gives the stress resultant forces and couples in the deformed plate:

$$N_{\alpha\beta} = \int_{-h^-}^{h^+} \sigma_{\alpha\beta}[(1+\varepsilon_\alpha)(1+\varepsilon_3)\cos\gamma_{\alpha 3}]\mathrm{d}z, \quad M_{\alpha\beta} = \int_{-h^-}^{h^+} \sigma_{\alpha\beta}\frac{z}{1+\varphi_3^{(1)}}[(1+\varepsilon_\alpha)(1+\varepsilon_3)^2\cos\gamma_{\alpha 3}]\mathrm{d}z, \tag{32}$$

where $N_{\alpha\beta}$ is the membrane forces and $M_{\alpha\beta}$ is the bending and twisting moments per unit length and $\{\alpha,\beta\} = \{1,2\}$. If the Kirchhoff assumption ($\varepsilon_3 = 0$) is used, Eq. (32) reduces to the form as in the textbook. The transverse shear forces are derived from the equations of equilibrium. The force balances based on the deformed middle surface in the three directions of Lagrangian coordinates, and the use of Eq. (15), give

$$\left[\frac{N_{\alpha\beta}\left(\delta_I^\beta + u_{I,\beta}^{(0)}\right)}{1+\varepsilon_\beta^{(0)}} + \frac{Q_\alpha\left(\delta_I^3 + \varphi_I^{(1)}\right)}{1+\varepsilon_3^{(0)}}\right]_{,\alpha} + q_I = \rho u_{I,tt}^{(0)} + I_z\varphi_{I,tt}^{(1)} \quad (\text{summation on } \alpha,\beta), \tag{33}$$

and the balances of moments in the Lagrangian coordinates, and the use of Eq. (15), give

$$\begin{aligned}&\left[\frac{M_{\alpha\beta}\left(\delta_{\gamma+1}^{\beta+1} + u_{\gamma+1,\beta+1}^{(0)}\right)}{1+\varepsilon_{\beta+1}^{(0)}}\right]_{,\alpha} + N_{12}\frac{(-1)^{\alpha+\gamma}w_{,\alpha+\gamma}^{(0)}\left(\delta_\gamma^\alpha + u_{\gamma,\alpha}^{(0)}\right)}{1+\varepsilon_{\gamma+1}^{(0)}}\\ &\quad - Q_\gamma\left[\frac{\left(1+\varphi_3^{(1)}\right)\left(\delta_\gamma^\alpha + u_{\gamma,\gamma}^{(0)}\right) - \varphi_\gamma^{(1)}u_{3,\gamma}^{(0)}}{1+\varepsilon_3^{(0)}}\right] + m_\gamma = I_z u_{\gamma,tt}^{(0)} + J_z\varphi_{\gamma,tt}^{(1)} \quad (\text{no summation on } \gamma),\end{aligned} \tag{34}$$

$$\left[\frac{M_{\alpha\beta}u_{3,\beta+1}^{(0)}}{1+\varepsilon_{\beta+1}^{(0)}}\right]_{,\alpha} + \left[\frac{N_{\alpha\beta}\left(\delta_\alpha^\beta + u_{\alpha,\beta}^{(0)}\right)}{1+\varepsilon_\beta^{(0)}} + \frac{Q_{\alpha+1}\varphi_\alpha^{(1)}}{1+\varepsilon_3^{(0)}}\right]\left(1+u_{\alpha+1,\alpha+1}^{(0)}\right) = 0, \tag{35}$$

where $I_z = \int_{-h^-}^{h^+}\rho_0 z\,\mathrm{d}z,\ J_z = \int_{-h^-}^{h^+}\rho_0 z^2\,\mathrm{d}z$. For any Greek symbol index $v \in \{\alpha,\beta,\gamma\} = \{1,2\}$, $v+1$ becomes $v-1$ if $v+1 > 2$. Therefore, the balances of equilibrium for thin plates give Eqs. (33)–(35). They together with Eqs. (25)–(28) constitute this approximate nonlinear theory for thin plates.

The equilibrium equations are based on the deformed middle surface in the Lagrangian coordinates. The alternative approach presented in Wempner (1973) can derive the similar equilibrium equations. With Eq. (32)–(35) can also be derived by using Boussinesq–Kirchhoff equations in Guo (1980).

3. Reduction to established theories

3.1. Kirchhoff plate theory

The Kirchhoff assumptions specify $\varepsilon_3 = \gamma_{\alpha 3} = 0$. From Eqs. (20) and (21), these constraints become

$$\left(\delta_I^3 + u_{I,3}\right)\left(\delta_I^3 + u_{I,3}\right) = 1, \tag{36}$$

$$\left(\delta_\alpha^{Ii} + u_{I,\alpha}\right)\left(\delta_3^I + u_{I,3}\right) = 0. \tag{37}$$

Substitution of Eq. (22) into Eqs. (36) and (37), expansion of them in Taylor series in z and the vanishing of the zero-order terms in z give

$$\left(\delta_I^3 + \varphi_I^{(1)}\right)\left(\delta_I^3 + \varphi_I^{(1)}\right) = 1, \tag{38}$$

$$\left(\delta_\alpha^I + u_{I,\alpha}\right)\left(\delta_3^I + \varphi_I^{(1)}\right) = 0. \tag{39}$$

$\varphi_I^{(1)}$ can be obtained from Eqs. (38) and (39) first. From the Taylor series, vanishing of the first-order terms in z gives three equations in $\varphi_I^{(2)}$ similar to Eqs. (38) and (39). The three equations plus $\varphi_I^{(1)}$ give $\varphi_I^{(2)}$; $\varphi_I^{(n)}$ for

$n = 3, 4, \ldots$ can be determined in a similar manner. With the first-order approximation in Eqs. (36) and (37), the solution to Eqs. (38) and (39) with $\{1 \equiv x,\ 2 \equiv y,\ 3 \equiv z\}$ and $\{u_1 \equiv u,\ u_2 \equiv v,\ u_3 \equiv w\}$ is

$$\varphi_1 = \frac{v_{,x}^{(0)} w_{,y}^{(0)} - \left(1 + v_{,y}^{(0)}\right) w_{,x}^{(0)}}{\sqrt{\left[v_{,x}^{(0)} w_{,y}^{(0)} - \left(1 + v_{,y}^{(0)}\right) w_{,x}^{(0)}\right]^2 + \left[u_{,y}^{(0)} w_{,x}^{(0)} - \left(1 + u_{,x}^{(0)}\right) w_{,y}^{(0)}\right]^2 + \left[\left(1 + u_{,x}^{(0)}\right)\left(1 + v_{,y}^{(0)}\right) - w_{,x}^{(0)} w_{,y}^{(0)}\right]^2}}, \tag{40}$$

$$\varphi_2 = \frac{u_{,y}^{(0)} w_{,x}^{(0)} - \left(1 + u_{,x}^{(0)}\right) w_{,y}^{(0)}}{\sqrt{\left[v_{,x}^{(0)} w_{,y}^{(0)} - \left(1 + v_{,y}^{(0)}\right) w_{,x}^{(0)}\right]^2 + \left[u_{,y}^{(0)} w_{,x}^{(0)} - \left(1 + u_{,x}^{(0)}\right) w_{,y}^{(0)}\right]^2 + \left[\left(1 + u_{,x}^{(0)}\right)\left(1 + v_{,y}^{(0)}\right) - w_{,x}^{(0)} w_{,y}^{(0)}\right]^2}}, \tag{41}$$

$$\varphi_3 = \frac{\left(1 + u_{,x}^{(0)}\right)\left(1 + v_{,y}^{(0)}\right) - w_{,x}^{(0)} w_{,y}^{(0)}}{\sqrt{\left[v_{,x}^{(0)} w_{,y}^{(0)} - \left(1 + v_{,y}^{(0)}\right) w_{,x}^{(0)}\right]^2 + \left[u_{,y}^{(0)} w_{,x}^{(0)} - \left(1 + u_{,x}^{(0)}\right) w_{,y}^{(0)}\right]^2 + \left[\left(1 + u_{,x}^{(0)}\right)\left(1 + v_{,y}^{(0)}\right) - w_{,x}^{(0)} w_{,y}^{(0)}\right]^2}} - 1, \tag{42}$$

where the superscript "(1)" has been dropped for notational convenience. Substitution of Eqs. (39)–(42) into Eqs. (25) and (27) gives

$$\varepsilon_\alpha \approx \varepsilon_\alpha^{(0)} + \frac{\left(\delta_i^\alpha + u_{i,\alpha}^{(0)}\right)\varphi_{i,\alpha}}{\left(1 + \varepsilon_\alpha^{(0)}\right)} z + \left\{ \frac{\varphi_{i,\alpha}\varphi_{i,\alpha}}{1 + \varepsilon_\alpha^{(0)}} - \frac{\left[\left(\delta_i^\alpha + u_{i,\alpha}^{(0)}\right)\varphi_{i,\alpha}\right]^2}{\left(1 + \varepsilon_\alpha^{(0)}\right)^3} \right\} z^2 + \cdots, \tag{43}$$

$$\gamma_{12} \approx \gamma_{12}^{(0)} + \frac{1}{\cos \gamma_{12}^{(0)}} \left\{ \frac{\left(\delta_i^1 + \varphi_{i,1}\right)\left(\delta_i^2 + \varphi_{i,2}\right)}{\left(1 + \varepsilon_1^{(0)}\right)\left(1 + \varepsilon_2^{(0)}\right)} - \sin \gamma_{12}^{(0)} \left[\frac{\left(\delta_i^1 + u_{i,1}^{(0)}\right)\varphi_{i,1}}{\left(1 + \varepsilon_1^{(0)}\right)^2} + \frac{\left(\delta_i^2 + u_{i,2}^{(0)}\right)\varphi_{i,2}}{\left(1 + \varepsilon_2^{(0)}\right)^2} \right] \right\} z + \cdots \tag{44}$$

Substitution of Eq. (39) in Eqs. (33)–(35) yields the equilibrium balance for Kirchhoff's plates.

3.2. Moderately large deflection of thin plates

For moderately large transverse deflection, assumptions for the middle surface are:

$$u_{1,\alpha}^{(0)} \approx o\left(u_{2,\alpha}^{(0)}\right) \approx o\left[\left(w_{,\alpha}^{(0)}\right)^2\right] \ll 1, \qquad 1 + \varepsilon_\alpha^{(0)} \approx 1. \tag{45}$$

Therefore, the strains of the middle surface become

$$\varepsilon_\alpha^{(0)} \approx u_{\alpha,\alpha}^{(0)} + \tfrac{1}{2}\left(w_{,\alpha}^{(0)}\right)^2, \qquad \gamma_{12}^0 \approx u_{2,1}^{(0)} + u_{1,2}^{(0)} + w_{,1}^{(0)} w_{,2}^{(0)}. \tag{46}$$

Substitution of Eqs. (45) and (46) into Eqs. (40)–(42) generates

$$\varphi_\alpha \approx -w_{,\alpha}^{(0)}, \quad \varphi_3 \approx 0. \tag{47}$$

Substitution of Eqs. (45)–(47) into Eqs. (30), (43) and, (44), and retention of terms that are first order in z, leads to

$$\varepsilon_\alpha \approx u^{(0)}_{\alpha,\alpha} + \frac{1}{2}\left(w^{(0)}_{,\alpha}\right)^2 - w^{(0)}_{,\alpha\alpha}z, \qquad \gamma_{12} \approx u^{(0)}_{2,1} + u^{(0)}_{1,2} + w^{(0)}_{,1}w^{(0)}_{,2} - 2w^{(0)}_{,12}z. \tag{48}$$

From Eqs. (38), (45) and (47), the equilibrium balances in Eqs. (33)–(35) give

$$\left[N_{\alpha\beta} - Q_\beta w^{(0)}_{,\alpha}\right]_{,\beta} + q_\alpha = \rho u^{(0)}_{\alpha,tt} - I w^{(0)}_{,\alpha tt}, \tag{49}$$

$$[N_{\alpha\beta}w^{(0)}_{,\beta} + Q_\alpha]_{,\alpha} + q_3 \approx \rho w^{(0)}_{,tt}, \tag{50}$$

$$M_{\alpha\beta,\beta} + N_{12}w^{(0)}_{,\alpha+1} - Q_\alpha + m_\alpha \approx I u^{(0)}_{\alpha,tt} - J w^{(0)}_{,\alpha tt}. \tag{51}$$

With density ρ_0 constant and $h^- = h^+ = h/2$, we have $I_z = 0$, $J_z = \rho_0 h^3/12$. When the rotary inertia is neglected, Eq. (51) becomes

$$M_{\alpha\beta,\beta} + N_{12}w^{(0)}_{,\alpha+1} - Q_\alpha + m_\alpha \approx 0, \tag{52}$$

when $m_\alpha = q_\alpha = 0$, the shear Q_α in Eq. (49) and N_{12} in Eq. (52) vanish, and Eqs. (49), (50) and (52) at $h^+ = h^- = h/2$ reduce to the nonlinear plate theory of Herrmann (1955).

The von Karman plate theory is recovered by letting $m_\alpha = q_\alpha = 0$ and neglecting the terms $w^{(0)}_{,\alpha} \ll 1$ in Eqs. (49) and (52):

$$N_{\alpha\beta,\beta} \approx 0, \tag{53}$$

$$M_{\alpha\beta,\beta} - Q_\alpha \approx 0. \tag{54}$$

Substitution of Eqs. (53) and (54) into Eq. (50) gives

$$N_{\alpha\beta}w^{(0)}_{,\alpha\beta} + M_{\alpha\beta,\alpha\beta} + q_3 = \rho w^{(0)}_{,tt}, \tag{55}$$

where the von Karman theory is applicable to plates of the moderately large deflection and small rotation, the theory in Eqs. (48)–(51) is applicable to plates of moderately large deflection and rotation because of Eq. (45).

3.3. Linear plate theory

The linear theory for thin plates is recovered from Eqs. (25)–(28) and Eqs. (33)–(35), when the Kirchhoff constraints are imposed: the elongation and shear in the plate are small compared to unity; the rotations are negligible compared to the elongation and shear:

$$u^{(0)}_{1,\alpha} \approx o(u^{(0)}_{2,\alpha}) \approx o(w^{(0)}_{,\alpha}) \ll 1, \qquad 1 + \varepsilon^{(0)}_\alpha \approx 1. \tag{56}$$

From the foregoing, the strains of the middle surface become

$$\varepsilon^{(0)}_\alpha \approx u^{(0)}_{\alpha,\alpha}, \qquad \gamma^{(0)}_{12} \approx u^{(0)}_{2,1} + u^{(0)}_{1,2}. \tag{57}$$

Substitution of Eqs. (56) and (57) into Eqs. (39)–(42) generates the rotation angles given by Eq. (47). With Eqs. (47), (56) and (57), Eqs. (33)–(35) reduce to the linear plate theory

$$N_{\alpha\beta,\beta} + q_\alpha \approx \rho u^{(0)}_{\alpha,tt}, \qquad M_{\alpha\beta,\alpha\beta} + q_3 \approx \rho w^{(0)}_{,tt}. \tag{58}$$

The shear forces are given by Eq. (54).

4. Applications

4.1. Membrane forces in a rectangular plate

Consider a thin, simply supported, rectangular plate subjected to a distributed transverse surface load q. The length, width and thickness of the plate are l, b and h, respectively. When Hooke's law for linearly elastic, isotropic materials is used, from Eq. (32) and Eqs. (48)–(51), the equations governing the equilibrium state for a nonlinear, isotropic rectangular plate at $m_\alpha = q_\alpha = 0$, $q_3 = q$, are

$$u_{,xx} + \frac{1-\mu}{2}u_{,yy} + \frac{1+\mu}{2}v_{,xy} + w_{,x}\left(w_{,xx} + \frac{1-\mu}{2}w_{,yy}\right) + \frac{1+\mu}{2}w_{,xy}w_{,y} + \frac{h^2}{12}\left\{\left[(\nabla^2 w)_{,x}w_{,x}\right]_{,x} + \left[(\nabla^2 w)_{,y}w_{,x}\right]_{,y}\right\} \approx 0, \tag{59}$$

$$v_{,yy} + \frac{1-\mu}{2}v_{,xx} + \frac{1+\mu}{2}u_{,xy} + w_{,y}\left(w_{,yy} + \frac{1-\mu}{2}w_{,xx}\right) + \frac{1+\mu}{2}w_{,xy}w_{,x} + \frac{h^2}{12}\left\{\left[(\nabla^2 w)_{,y}w_{,y}\right]_{,y} + \left[(\nabla^2 w)_{,x}w_{,y}\right]_{,x}\right\} \approx 0, \tag{60}$$

$$\frac{h^2}{12}\nabla^4 w - \left\{\left[\left(u_{,x} + \frac{1}{2}(w_{,x})^2\right) + \mu\left(v_{,y} + \frac{1}{2}(w_{,y})^2\right)\right]w_{,x} + (1-\mu)(v_{,x} + u_{,y} + w_{,x}w_{,y})w_{,y}\right\}_{,x} - \left\{\left[\left(v_{,y} + \frac{1}{2}(w_{,y})^2\right) + \mu\left(u_{,x} + \frac{1}{2}(w_{,x})^2\right)\right]w_{,y} + (1-\mu)(v_{,x} + u_{,y} + w_{,x}w_{,y})w_{,x}\right\}_{,y} \approx \frac{(1-\mu^2)q}{Eh}, \tag{61}$$

where ρ_0, E and μ are density, Young's modulus and Poisson's ratio, respectively. The superscript "(0)" has been dropped for notational convenience. The h^2 terms in Eqs. (59) and (60) and the $(1-\mu)$ terms in Eq. (61) that are ignored in the von Karman theory are the contributions of normal and shear forces in the in-plane.

With Eq. (32), Eqs. (45)–(48), application of the Hooke's law to linear, isotropic, elastic materials gives membrane forces N_x, N_y, N_{xy}, i.e.,

$$\left.\begin{aligned} N_x &= \tfrac{Eh}{1-\mu^2}\left[\left(u_{,x} + \tfrac{1}{2}(w_{,x})^2\right) + \mu\left(v_{,y} + \tfrac{1}{2}(w_{,y})^2\right)\right], \\ N_y &= \tfrac{Eh}{1-\mu^2}\left[\left(v_{,y} + \tfrac{1}{2}(w_{,y})^2\right) + \mu\left(u_{,x} + \tfrac{1}{2}(w_{,x})^2\right)\right], \\ N_{xy} &= \tfrac{Eh}{2(1+\mu)}(v_{,x} + u_{,y} + w_{,x}w_{,y}) \end{aligned}\right\} \tag{62}$$

and the boundary conditions are

$$\left.\begin{aligned} &u(0,y) = u(l,y) = w(0,y) = w(l,y) = 0, \quad w_{,xx}(0,y) = w_{,xx}(l,y) = 0; \\ &v(x,0) = v(x,b) = w(x,0) = w(x,b) = 0, \quad w_{,yy}(x,0) = w_{,yy}(x,b) = 0 \end{aligned}\right\}. \tag{63}$$

A transverse displacement in Eq. (61), satisfying the displacement boundary conditions of Eq. (63), can be represented by

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} h f_{mn} \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{64}$$

where f_{mn} is unknown. To compare with existing solutions given by the von Karman theory, a single mode (m, n) is considered here. Following the procedure of Chu and Herrmann (1956), one obtains the solutions for u *and* v from Eqs. (59), (60), (63) and (64) as

$$\left.\begin{aligned} u &\approx \frac{h^2}{l}\frac{m\pi}{16}f_{mn}^2 \sin\left(\frac{2m\pi x}{l}\right)\left[\cos\left(\frac{2n\pi y}{b}\right) - \left(1 - \frac{\mu(nl)^2}{(mb)^2}\right) + \left(\frac{h}{l}\right)^2\frac{(m\pi)^2}{6}\left(1 + \frac{(nl)^2}{(mb)^2}\right)\sin^2\left(\frac{n\pi y}{b}\right)\right], \\ v &\approx \frac{h^2}{b}\frac{n\pi}{16}f_{mn}^2 \sin\left(\frac{2n\pi y}{b}\right)\left[\cos\left(\frac{2m\pi x}{l}\right) - \left(1 - \frac{\mu(mb)^2}{(nl)^2}\right) + \left(\frac{h}{b}\right)^2\frac{(n\pi)^2}{6}\left(1 + \frac{(mb)^2}{(nl)^2}\right)\sin^2\left(\frac{m\pi x}{l}\right)\right] \end{aligned}\right\} \tag{65}$$

where f_{mn} can be predicted in Eq. (61) through the Galerkin method. For the von Karman theory, the displacement in Chu and Herrmann (1956) is

$$\left.\begin{aligned} u &\approx \frac{h^2}{l}\frac{m\pi}{16}f_{mn}^2 \sin\left(\frac{2m\pi x}{l}\right)\left[\cos\left(\frac{2n\pi y}{b}\right) - \left(1 - \frac{\mu(nl)^2}{(mb)^2}\right)\right], \\ v &\approx \frac{h^2}{b}\frac{n\pi}{16}f_{mn}^2 \sin\left(\frac{2n\pi y}{b}\right)\left[\cos\left(\frac{2m\pi x}{l}\right) - \left(1 - \frac{\mu(mb)^2}{(nl)^2}\right)\right]. \end{aligned}\right\} \tag{66}$$

The terms in Eq. (65), not appearing in Eq. (66) arise from the shear forces in the longitudinal directions. When $\pi^2 h^2[(mb)^2 + (nl)^2] \ll 6(bl)^2$, the additional terms can be neglected. When the thickness is very small compared to the width and the mode number is very small, Eqs. (65) and (66) can give $u \approx v \approx 0$ by the linear theory for $f_{mn} \ll 1$. The $(h/b)^2$ and $(h/l)^2$ terms in Eq. (65) become significant as the thickness increases and when m, n become large.

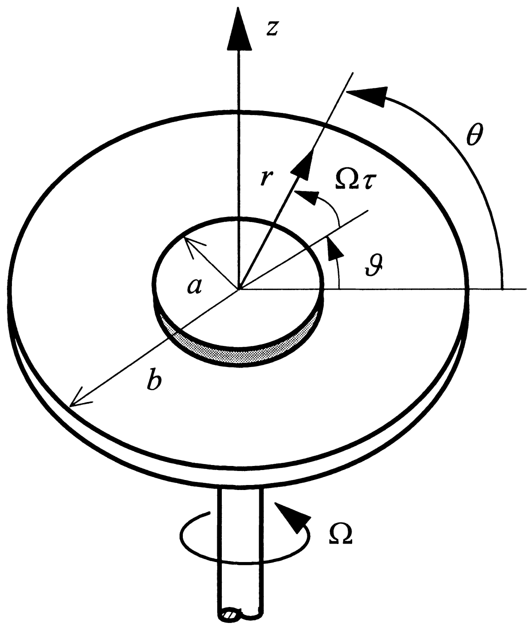

Fig. 3. Rotating disk with fully clamped hub.

4.2. Equilibrium solution for rotating disk

Consider the uniform, flexible, circular disk rotating with constant angular speed Ω as shown in Fig. 3. The coordinate (r, ϑ, t) rotates with the disk and (r, θ, t) remains fixed in space. They satisfy

$$\theta = \vartheta + \Omega t. \tag{67}$$

Consider equilibrium solutions of the plate undergoing large amplitude displacement. Neglect of the in-plane inertia in Eqs. (49)–(51), elimination of time dependence, and transformation to polar coordinates give

$$u_{r,rr} + \frac{u_{r,r}}{r} - \frac{u_r}{r^2} + \frac{1-\mu}{2r^2}u_{r,\theta\theta} + \frac{1+\mu}{2r}u_{\theta,r\theta} - \frac{3-\mu}{2r^2}u_{\theta,\theta} + u_{z,r}\left(u_{z,rr} + \frac{1-\mu}{2r^2}u_{z,\theta\theta}\right) + \frac{1+\mu}{2r^2}u_{z,r\theta}u_{z,\theta}$$
$$+ \frac{1-\mu}{2r}\left[(u_{z,r})^2 - \frac{1}{r^2}(u_{z,\theta})^2\right] + \frac{h^2}{12}\left\{\left[(\nabla^2 u_z)_{,r}u_{z,r}\right]_{,r} + \frac{1}{r^2}\left[(\nabla^2 u_z)_{,\theta}u_{z,r}\right]_{,\theta}\right\} + \frac{\rho_0(1-\mu^2)}{E}\Omega^2 r = 0, \tag{68}$$

$$\frac{u_{\theta,\theta\theta}}{r^2} + \frac{1-\mu}{2}\left(u_{\theta,rr} + \frac{u_{\theta,r}}{r} - \frac{u_\theta}{r^2}\right) + \frac{1+\mu}{2r}u_{r,r\theta} + \frac{3-\mu}{2}u_{r,\theta} + \frac{1+\mu}{2r}u_{z,r}u_{z,r\theta}$$
$$+ \frac{u_{z,\theta}}{r}\left[\frac{u_{z,\theta\theta}}{r^2} + \frac{1-\mu}{2}\left(u_{z,rr} + \frac{u_{z,r}}{r}\right)\right] + \frac{h^2}{12}\left\{\left[\frac{1}{r}(\nabla^2 u_z)_{,r}u_{z,\theta}\right]_{,r} + \frac{1}{r^3}\left[(\nabla^2 u_z)_{,\theta}u_{z,\theta}\right]_{,\theta}\right\} = 0 \tag{69}$$

$$\frac{1}{r}\left(r\left\{u_{r,r} + \frac{(u_{z,r})^2}{2} + \mu\left[\frac{u_r}{r} + \frac{u_{\theta,\theta}}{r} + \frac{(u_{z,\theta})^2}{2r^2}\right]\right\}u_{z,r} + (1-\mu)\left[u_{\theta,r} + \frac{u_{r,\theta}}{r} - \frac{u_\theta}{r} + \frac{u_{z,r}u_{z,\theta}}{r}\right]u_{z,\theta}\right)_{,r}$$
$$+ \frac{1}{r}\left(\frac{1}{r}\left\{\frac{u_r}{r} + \frac{u_{\theta,\theta}}{r} + \frac{(u_{z,\theta})^2}{2r^2} + \mu\left[u_{r,r} + \frac{(u_{z,r})^2}{2}\right]\right\}u_{z,\theta} + (1-\mu)\left[u_{\theta,r} + \frac{u_{r,\theta}}{r} - \frac{u_\theta}{r} + \frac{u_{z,r}u_{z,\theta}}{r}\right]u_{z,r}\right)_{,\theta}$$
$$+ \frac{(1-\mu^2)q}{Eh} = \frac{h^2}{12}\nabla^4 w + \frac{\rho_0(1-\mu^2)}{E}\Omega^2 u_{z,\theta\theta}. \tag{70}$$

If the h^2 terms in Eqs. (68) and (69) and the $(1-\mu)$ terms in Eq. (70) are neglected, Eqs. (68)–(70) reduce to the von Karman equations, as presented in Nowinski (1964) for equilibrium of the rotating disk. Such terms in Eqs. (68)–(70) are the contributions of normal and shear force in the in-plane. When the Hooke's law for linear, isotropic, elastic materials is used, the membrane forces $\{N_r, N_\theta, N_{r\theta}\}$ through Eqs. (45)–(47) are:

$$\left.\begin{aligned} N_r &= \frac{Eh}{1-\mu^2}\left\{u_{r,r} + \frac{1}{2}(u_{z,r})^2 + \mu\left[\frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2r^2}(u_{z,\theta})^2\right]\right\}, \\ N_\theta &= \frac{Eh}{1-\mu^2}\left\{\frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2r^2}(u_{z,\theta})^2 + \mu\left[u_{r,r} + \frac{1}{2}(u_{z,r})^2\right]\right\}, \\ N_{r\theta} &= \frac{Eh}{2(1+\mu)}\left(u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{1}{r}u_\theta + \frac{1}{r}(u_{z,r})(u_{z,\theta})\right) \end{aligned}\right\}. \tag{71}$$

The boundary conditions are

$$\left.\begin{array}{lr} u_r = u_\theta = u_z = 0, \quad u_{z,r} = 0 & \text{at } r = a \\ u_{z,rr} + \mu(\frac{1}{r}u_{z,r} + \frac{1}{r^2}u_{z,\theta\theta}) = 0, \quad (\nabla^2 u_z)_{,r} + \frac{1-\mu}{r^2}(u_{z,r} - \frac{1}{r}u_z)_{,\theta\theta} = 0 & \text{at } r = b \end{array}\right\} \tag{72}$$

and the radial and shear forces at $r = b$ satisfying $N_r = N_{r\theta} = 0$ give

$$u_{r,r} + \frac{1}{2}(u_{z,r})^2 + \mu\left[\frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2r^2}(u_{z,\theta})^2\right] = 0, \qquad u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{1}{r}u_\theta + \frac{1}{r}(u_{z,r})(u_{z,\theta}) = 0. \tag{73}$$

Let the modal transverse load be

$$q = q_c\cos(s\theta) + q_s\sin(s\theta). \tag{74}$$

and an approximate solution satisfying the boundary condition (72) be

$$u_z = h\sum_{m=0}^{4} C_m\left(\frac{r}{b}\right)^{m+s}[f_{sc}\ \cos(s\theta) + f_{ss}\ \sin(s\theta)]. \tag{75}$$

Substitution of Eq. (75) into Eqs. (68) and (69) and their solutions into Eqs. (72) and (73) give modal u_r, u_θ. Substitution of u_r, u_θ, u_z in Eq. (71) predicts the approximate equilibrium membrane forces. Finally, application of the Galerkin method to Eq. (70) generates the approximate equilibrium solution.

The nondimensionalized deflection and transverse load amplitudes are:

$$A = \sqrt{f_{sc}^2 + f_{ss}^2}, \quad Q = \frac{q_0(1-\mu^2)\alpha}{\beta E}, \quad q_0 = \sqrt{q_c^2 + q_s^2}, \tag{76}$$

where

$$\alpha = \sum_{m=0}^{4}\frac{1}{m+s+1}(1-\kappa^{m+s+1})C_m, \quad \beta = \frac{1}{12}\sum_{m=0}^{4}\sum_{n=0}^{4}\frac{1}{m+n+2s+1}\left(1-\kappa^{m+n+2s+1}\right)C_mC_n. \tag{77}$$

Note that $\kappa = a/b$ is the clamping ratio. Coefficients C_m are determined through satisfaction of the boundary condition, and coefficients α and β are generated by application of the Galerkin's method to Eq. (70).

Comparisons of predictions of the equilibrium amplitude A for load amplitude Q by this theory, the von Karman theory and the linear theory are illustrated in Figs. 4–6 for a model of a 3.5 in. diameter, computer memory hard disk with inner and outer radii $a = 15.5$ mm and $b = 43$ mm, respectively. This theory and the von Karman theory give identical prediction for $s = 0$ in Fig. 4 because of the symmetric loading and response. The relative errors in the transverse load between the linear and nonlinear theories at $A = 1.0$ are about 12.2% and 9.8% for $\Omega = 0$ and 20k$^-$ rpm. For a specified $Q = 50$ at $s = 0$, the relative errors for the displacement amplitudes between the linear and nonlinear theories are less than 8% and 5% for $\Omega = 0$ and 20k$^-$ rpm. When $s \neq 0$, the transverse load and response of the disk are asymmetric. For $s = 1$ in Fig. 5, predictions by the two nonlinear theories are close when $A \leqslant 0.2$, and their relative error in the transverse load between the nonlinear theories is less than 4%. Predictions of the linear theory are close to the nonlinear theories when $A \leqslant 0.1$ and the relative error of the linear theory to the theory is less than 2.5%. Similar to the discussion for $s = 0$, for a specified $Q = 20$ at $s = 1$, the relative errors of the displacement amplitudes of the linear theory and the von Karman theory to the proposed theory are about 20% and 8% at $\Omega = 0$. For $s = 4$ in Fig. 6, the three theories, with the maximum relative error less than 1%, are in good

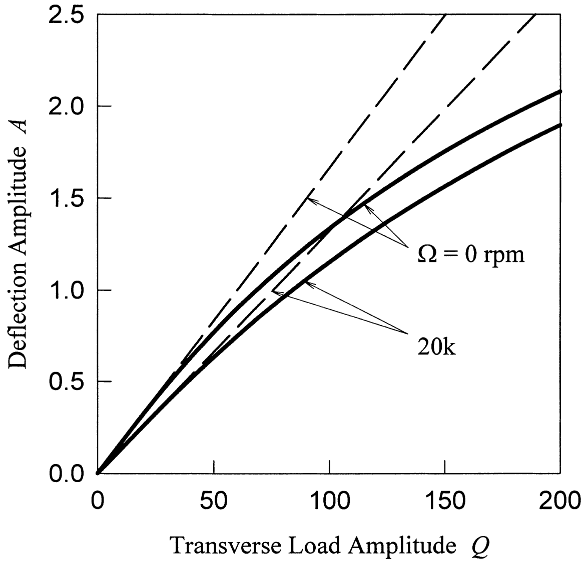

Fig. 4. The equilibrium deflection versus transverse load ($s = 0$). The solid, dot-dash and dash lines denote the disk displacements predicted by this theory, the von Karman theory and the linear theory ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 2770$ kg/m^3, $E = 73$ GPa, $\mu = 0.33$).

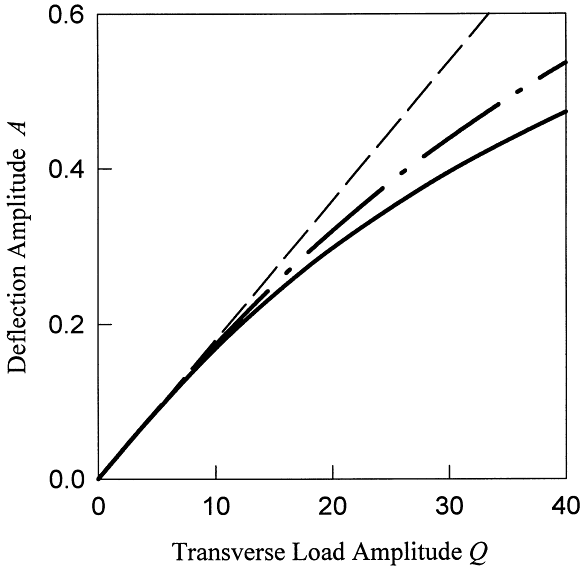

Fig. 5. The equilibrium deflection versus transverse load ($s = 1$) at $\Omega = 0$. The solid, dot-dash and dash lines denote the disk displacements predicted by this theory, the von Karman theory and the linear theory ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 2770$ kg/m^3, $E = 73$ GPa, $\mu = 0.33$).

agreement when $A \leqslant 0.1$. When $A \leqslant 0.2$, the linear theory is in agreement with the proposed theory with less than 3% relative error. The von Karman theory gives results poorer than the linear theory because the von Karman theory models a balance of forces created by the curvature of the disks only in the transverse

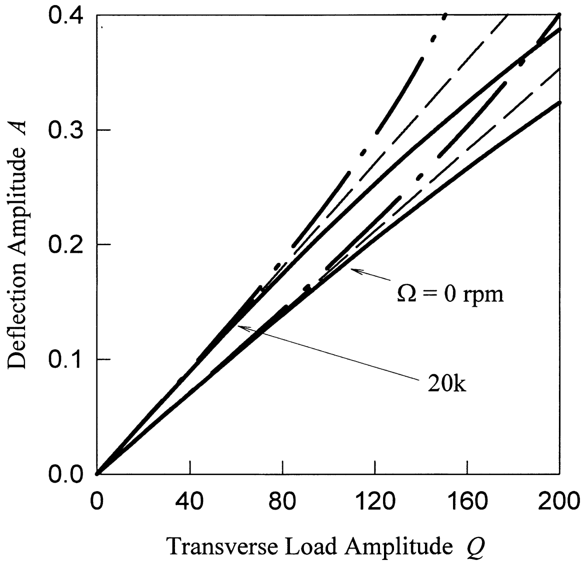

Fig. 6. The equilibrium deflection versus transverse load ($s = 4$). The solid, dot-dash and dash lines denote the disk displacements predicted by this theory, the von Karman theory and the linear theory ($a = 15.5$ mm, $b = 43$ mm, $h = 0.775$ mm, $\rho_0 = 2770$ kg/m^3, $E = 73$ GPa, $\mu = 0.33$).

direction. The membrane forces arising from force balances in the in-plane directions and moment balances are not included. For the symmetric response, the von Karman theory is identical to the proposed theory. For asymmetric responses, two nonlinear theories will give different predictions. Especially, for $s = 4$, the von Karman theory gives the response showing the *softening*-spring behavior but the proposed theory gives the response showing the *hardening*-spring behavior. The softening-spring behavior of the rotating disks indicates that the corresponding stiffness becomes small when the external loading increases. However, for the hardening-spring behavior, the stiffness of such rotating disks increases with increasing the external loading. Predictions of displacement of the disk produced by the three plate theories deviate when the disk deflection and/or the nodal diameter s become large. Agreement of those predictions improves as Ω increases.

5. Conclusions

An approximate theory for geometrically nonlinear plates is developed in this article. The theory includes the physical strains and equilibrium equations of the plates based on the exact geometry of the deformed middle surface in the Lagrangian coordinates. Limitations on the constitutive laws are not required because the physical strains instead of the Lagrangian and Eulerian strains are used to develop such approximate theory. Therefore, the geometrical relations and equilibrium equations presented in the proposed theory are applicable for all deformation processes of thin plates, such as elasticity, plasticity and the others.

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